ECE/MAT 211A Mid-Term exam

Wednesday Feb. 13th 2-4 pm, 2002.

You may use the following during the exam:

1. The five handouts that I have given out during class
2. Notes that you have made yourself during class and while studying. These may be handwritten or typed by yourself - extensive photocopies are not allowed.
3. Your own solutions to the homework problems. Photocopies of my solutions are not allowed.
4. A calculator

Constants, etc.

- The charge on an electron is 1.602×10^{-19} C.
- Avagadro's number is 6.023×10^{23} per mole.
- Boltzmann's constant, $k = 1.38066 \times 10^{-23}$ JK^{-1}.
- Planck's constant, $h = 6.626 \times 10^{-34}$ Js.
- The mass of an electron, $m_e = 9.1095 \times 10^{-31}$ kg.
- The proton mass, $u = 1.66 \times 10^{-27}$ kg.
- The speed of light, $c = 2.998 \times 10^8$ ms^{-1}.
- $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$
- $1 + \cos(x) = 2\cos^2(\frac{x}{2})$
- $1 - \cos(x) = 2\sin^2(\frac{x}{2})$
- $\sin(x) = 2\sin(\frac{x}{2})\ cos(\frac{x}{2})$

The number of points for each part of each question is given in square brackets and bold type after each question. The number of points reflects the difficulty and number of components for each answer.

DISCLAIMER. This examination develops the concept of electron spin, however no prior knowledge of spin properties is necessary or assumed.

The operator S_z determines the projection of the spin of an electron along the z direction. The two eigenvalues of the S_z operator are $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$. Let's call the eigenstates corresponding to these eigenvalues $|\alpha>$ and $|\beta>$ respectively. (Often we refer to states $|\alpha>$ and $|\beta>$ as the 'up' and 'down' spin states.)

(a) The component of electron spin along the z direction is a measurable quantity. What does this tell us about the mathematical nature of the S_z operator and why? **[1]**

(b) Given that any possible spin state for an electron can be described by a linear combination of the states $|\alpha>$ and $|\beta>$ (i.e. the states $|\alpha>$ and $|\beta>$ form a complete set for electron spin), what can we say about the value of the inner product

$$< \alpha|\beta >$$

and why? **[1]**

(c) In the absence of a magnetic field, $|\alpha>$ and $|\beta>$ have the same energy. However if a magnetic field, B_z, is applied along the z direction, the Hamiltonian is

$$H_{\alpha,\alpha} = -\mu B_z \quad ; \quad H_{\alpha,\beta} = 0$$
$$H_{\beta,\alpha} = 0 \quad ; \quad H_{\beta,\beta} = \mu B_z$$

where μ is the magnetic moment of the electron, and $H_{i,j} =< i|H|j >$, etc.

What are the energy eigenvalues of an electron in a magnetic field along the z axis? Are the energy eigenstates in the presence of a field along the z axis the same or different from $|\alpha>$ and $|\beta>$? What does this tell us about the commutativity of H and S_z? **[3]**

(d) Explain what is meant by a stationary state. Are the states $|\alpha>$ and $|\beta>$ stationary states in this case? How do $|\alpha>$ and $|\beta>$ evolve with time in the presence of a magnetic field applied along z? **[3]**

(e) Now suppose that the electron is prepared in an initial state which is the normalized sum of the two eigenstates $|\alpha>$ and $|\beta>$, with equal coefficients. (In fact this state corresponds to an electron with its spin aligned along the $+x$ direction). At a later time, t, the wavefunction is given by

$$|\psi(t) >= C_\alpha(t)|\alpha > +C_\beta(t)|\beta >$$

What are the forms of $C_\alpha(t)$ and $C_\beta(t)$? What is the probablity that the electron will be found with its spin aligned along the $+x$ direction (i.e. in the same state as its initial state) at some later time t? Sketch the probablity of finding the spin aligned along $+x$ as a function of time. With what frequency does the probability repeat itself? On what parameters does the frequency depend. **[7]**

(f) The Hamiltonian for an electron in a *general* magnetic field, B, with components B_x, B_y and B_z along the x, y and z directions is

$$H_{\alpha,\alpha} = -\mu B_z \quad ; \quad H_{\alpha,\beta} = -\mu(B_x - iB_y)$$
$$H_{\beta,\alpha} = -\mu(B_x + iB_y) \quad ; \quad H_{\beta,\beta} = \mu B_z$$

Is the Hamiltonian Hermitian? (Explain why). What are the eigenvalues? Compare the eigenvalues in this case with those you obtained in part (c) and comment on any changes in the physics of the problem. **[4]**

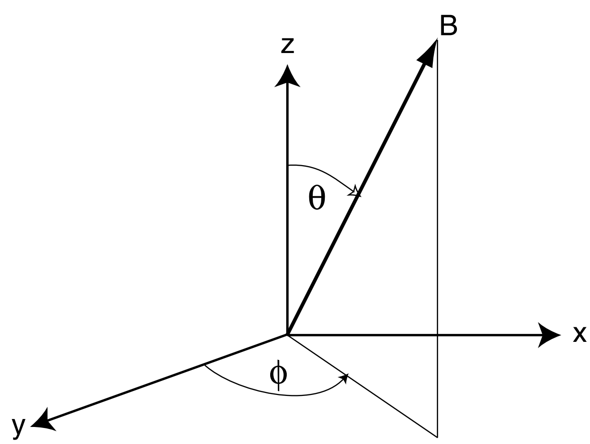
(g) Show that the eigenvectors for an electron described by the Hamiltonian in part (e), can be written as

$$\sin(\frac{\theta}{2})e^{-i\phi/2}|\alpha> \quad - \quad \cos(\frac{\theta}{2})e^{i\phi/2}|\beta>$$
$$\cos(\frac{\theta}{2})e^{-i\phi/2}|\alpha> \quad + \quad \sin(\frac{\theta}{2})e^{i\phi/2}|\beta>$$

Here θ is the polar angle between the z axis and the field B, and ϕ is the azimuthal angle, so that

$$B_x = B\sin\theta\cos\phi; \quad B_y = B\sin\theta\sin\phi; \quad B_z = B\cos\theta$$

as shown below:



[4]

Start the following on a new page - this homework will be divided between two graders!

(h) In fact the two state system of electron spin is so important that a new notation is often introduced to describe it, and the Hamiltonian is written as

$$H = -\mu(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z).$$

Show that this definition is consistent with that in part (f) if the σ_x, σ_y and σ_z are defined by the following 2×2 matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are known as the Pauli spin matrices. **[2]**

(i) Show that the column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenfunctions of the σ_z matrix. What are the eigenvalues? Make an argument that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (usually called 'spinors') are the same as $|\alpha>$ and $|\beta>$. In fact, our original operator, S_z is just $\frac{\hbar}{2}\sigma_z$, and we similarly define $S_x = \frac{\hbar}{2}\sigma_x$ and $S_y = \frac{\hbar}{2}\sigma_y$, where S_x and S_y are the matrices describing the projection of the spin of an electron along the x and y directions respectively. **[3]**

(j) Calculate the value of the commutator, $[S_x, S_y]$. **[2]**

(k) Given that an electron is in the state $|\alpha>$, calculate the uncertainty in its projection along the x and y directions respectively, i.e. calculate $\sqrt{(<S_x^2> - <S_x>^2)}$ and $\sqrt{(<S_y^2> - <S_y>^2)}$. **[4]**

(l) State the generalized uncertainty principle relating the product of the uncertainties of the expectation values of two operators to the value of their commutator. Based on your answers to parts (j) and (k), verify the uncertainty principle in this case. **[3]**