If $\sigma$, $\varepsilon$ vary with $x$, then we generalize to a local strain,

$$\varepsilon(x) = \frac{\partial u(x)}{\partial x}$$

$\Rightarrow$ (from Hooke's law) a local stress

$$\sigma(x) = E\,\frac{\partial u(x)}{\partial x} \quad ⊛$$

Now the force on a small element between $x$ and $x + dx$ is

$$[\sigma(x+dx) - \sigma(x)]A = \frac{\partial \sigma}{\partial x} A\,dx$$

from ⊛ $\rightarrow$ $= \dfrac{\partial^2 u}{\partial x^2} E\,A\,dx$

the element also has velocity $= \dfrac{\partial u}{dt}$

acceleration $= \dfrac{\partial^2 u}{\partial t^2}$

mass $= \rho A\,dx$ ($\rho$ = density)

$\therefore$ from $F = ma$, $\dfrac{\partial^2 u}{\partial x^2} EA\,dx = \dfrac{\partial^2 u}{\partial t^2} \rho A\,dx$

OR $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\rho}{E} \dfrac{\partial^2 u}{\partial t^2}$ the wave equation!

solutions are elastic waves of displacement:-

$$u(x,t) = u_0\,e^{-i(\omega t - kx)}$$

The elastic wave of displacements travels down the rod with velocity

$$V_s = \frac{\omega}{k} = \sqrt{\frac{E}{\rho}}$$

– this is the velocity of sound in the solid.

(Note that we've considered longitudinal waves)

Problem 7.2, page 118