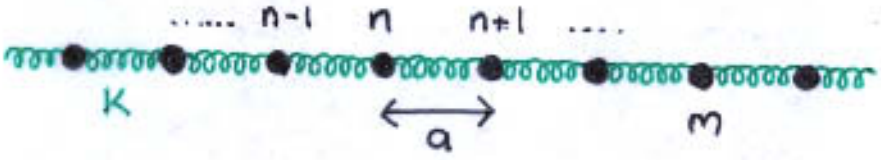
Vibrating Atoms - Dispersion

1D chain of atoms, mass M, spacing a, spring constant K.



If nth atom is displaced u_n from = m pos^n. then force on it from (n-1)th atom is $-K(u_n - u_{n-1})$
from (n+1)th atom is $-K(u_n - u_{n+1})$

so (*) $F = ma = -K(u_n - u_{n-1} + u_n - u_{n+1}) = -K(2u_n - u_{n-1} - u_{n+1})$

- a wave equation with a discrete variable!

Try the solution $u_n = u_0 e^{-i(\omega t - kna)}$
(na = m pos^n. of nth atom)

subst. in (*) $\Rightarrow$ $-M\omega^2 = -K[2 - e^{-ika} - e^{ika}]$

or $\omega = \left(\frac{4K}{M}\right)^{1/2} \left|\sin\left(\frac{ka}{2}\right)\right|$

i.e. soln. of (*) is

$u_n = u_0 e^{-i(\omega t - kna)}$ with $\omega = \sqrt{\frac{4K}{M}} \left|\sin\left(\frac{ka}{2}\right)\right|$

Low Frequency Limit, $k \to 0$, $\left|\sin\left(\frac{ka}{2}\right)\right| \to \frac{ka}{2}$ $\therefore$ $\omega \to ka\sqrt{\frac{K}{M}}$

so velocity = $\frac{\omega}{k} = a\sqrt{\frac{K}{M}}$ - a constant!
(sound waves are in this limit)

BUT in general the velocity depends on the wave number - this gives DISPERSION