1. Consider the set of monomial basis functions $\phi_n(x) = x^n$, $n = 0, 1, 2, \ldots, \infty$ over the interval $x \in [-1, 1]$. These form a complete basis in the Hilbert space $L_2(-1,1)$.

   (a) Using the $L_2$ inner product discussed in class, show by hand that the first two of these basis functions are linearly independent (LI). Discuss the orthogonality properties of the monomials.

   (b) Using Mathematica or another tool, show that the first five of these basis functions are LI.

   (c) Use the the Gram-Schmidt procedure to construct a set of orthogonal polynomials from the monomial basis up to $n = 3$. Show that the resulting polynomials are proportional to the Legendre polynomials, $P_n(x)$, and are identically $P_n(x)$ if the normalization $P_n(1) = 1$ is chosen. You should work out the first two polynomials by hand and can use Mathematica if you wish for the remainder.

   (d) Confirm for this normalization that $\|P_n\|^2 = 2/(2n + 1)$.

2. Reconsider the same monomial basis and interval $x \in [-1, 1]$ as in problem 1. In this problem we introduce a new Hilbert space $L_{2w}(-1,1)$, with inner product and norm

   $$(f, g) = \int_{-1}^{1} dx \, w(x) f^*(x) g(x), \quad ||f|| = (f, f)^{1/2}$$

   where $w(x) \geq 0$ is a so-called “weight” function.

   (a) Use the Gram-Schmidt procedure to construct a set of orthogonal polynomials in $L_{2w}(-1,1)$ with weight function $w(x) = 1/\sqrt{1-x^2}$ from the monomial basis up to $n = 3$. You may find the substitution $x = \cos \theta$ useful to simplify the necessary inner product integrals. Show that the resulting polynomials are proportional to the Chebyshev polynomials, $T_n(x)$, and are identically $T_n(x)$ if the normalization $T_n(1) = 1$ is chosen. You should work out the first two polynomials by hand and can use Mathematica if you wish for the remainder.

   (b) Confirm that your Chebyshev polynomials conform to the formula

   $$T_n(\cos \theta) = \cos(n\theta)$$

   This defining formula proves very important in numerical methods (“Chebyshev collocation”) for solving PDEs and boundary value problems in confined geometries.
3. Consider an orthonormal, complete basis \( \phi_1, \phi_2, \ldots \) in \( L_{2w}(a, b) \) for some general \( w(x) \geq 0 \). If two functions \( f(x) \) and \( g(x) \) in \( L_{2w} \) are expressed as generalized Fourier Series in the basis with coefficients \( b_i \) and \( c_i \), respectively, prove the Parseval relation

\[
(f, g) = \sum_{i=1}^{\infty} b_i^* c_i
\]

and as a special case, Bessel’s equality

\[
||f||^2 = \sum_{i=1}^{\infty} |b_i|^2
\]