1. Consider representing the function \( f(x) = x^2 \exp(x) \) in a Fourier series.

(a) Derive expressions for the coefficients appropriate for a Fourier series representation over the interval \( x \in [-2, 2] \).

(b) Repeat a) for the same function defined over \( x \in [0, 2] \) and making the even periodic extension to \([-2, 0]\).

(c) Repeat a) for the same function defined over \( x \in [0, 2] \) and making the odd periodic extension to \([-2, 0]\).

(d) Evaluating the coefficients numerically, compare the convergence properties of the even and odd extensions with the number of retained terms \( N \), visualizing the error on a log-log plot. Use as your error metric both the \( L_2 \) and \( L_1 \) norms of the difference between the exact function and the \( N \)-term Fourier series approximant. Are your results consistent with the convergence theorems discussed in class?

2. Consider the following problem of heat conduction in a one-dimensional beam with an insulated end and the other end subjected to convective cooling (note \( x \) and \( t \) subscripts denote partial derivatives):

\[
\begin{align*}
T_t &= \alpha T_{xx}, \quad t > 0, 0 < x < L \\
T_x(0, t) &= 0, \quad t > 0 \\
-kT_x(L, t) &= h[T(L, t) - T_1], \quad t > 0 \\
T(x, 0) &= T_1 \cos(x\frac{2\pi}{L}), \quad 0 < x < L
\end{align*}
\]

(a) Before starting to solve the problem, non-dimensionalize the \( x \) variable by scaling with \( L \), the \( t \) variable by scaling with \( L^2/\alpha \), and the temperature \( T \) by scaling with \( T_1 \). What non-dimensionless group “\( G \)” remains in the equations?

(b) Solve the problem by separation of variables, providing explicit formulas for any expansion coefficients that appear in your solution.

(c) Using Mathematica or another tool, generate plots of your dimensionless temperature profile at various dimensionless times, showing evolution to the steady state. Repeat for different values of \( G \).
3. Legendre’s differential equation for the interval \( x \in [-1, 1] \) is

\[
(1 - x^2) \frac{d^2 \phi}{dx^2} - 2x \frac{d \phi}{dx} + n(n + 1) \phi = 0
\]

(a) Classify this equation within the context of Sturm-Liouville theory. Are there singular points, and if yes, where?

(b) Use Mathematica to explore the solutions of this equation. What can you say about the difference between solutions when \( n \) is an integer versus when \( n \) is a real number, considering their behavior over the whole interval \( x \in [-1, 1] \)?

(c) For the special case of \( n = 0, 1, 2, \ldots \), what functions satisfy the equation that are finite at both \( x = \pm 1 \)? Look them up and plot some of them in Mathematica.