1. Consider a tubular heat exchanger immersed in a bath with varying “ambient” temperature around the tube, \( T_a(t) \). If the velocity in the tube is uniform at \( V \), a simple model for the temperature in the tube \( T(z, t) \) is obtained by balancing heat transfer by convection and film resistance. This leads to the following PDE, initial and boundary conditions:

\[
T_t + V T_z = H(T_a - T)
\]

\[
T(z, 0) = T_0(z), \quad 0 \leq z \leq L
\]

\[
T(0, t) = T_i(t), \quad t \geq 0
\]

where \( T_0(z) \) is the initial temperature distribution in the tube and \( T_i(t) \) is the inlet temperature of the fluid in the tube. Using the method of characteristics, solve this problem. Note that there is a particular characteristic in the \( x-t \) plane that divides regions where the initial temperature distribution is felt or not. You will need to develop separate solutions in the two regions.

2. The following equation describes the time evolution of the momentum distribution function, \( W(p, t) \), in the theory of Brownian motion:

\[
W_t = (pW)_p + W_{pp}
\]

Consider an initial value problem, where \( W(p, 0) = f(p) \) is a prescribed initial velocity distribution function for \(-\infty < p < \infty\). Adopting the boundary conditions at infinity \( W, W_p \to 0 \) for \( p \to \pm \infty \), solve this equation by first Fourier transforming over \( p \) to reduce the order of the PDE, then applying the method of characteristics, and finally back-transforming.

3. Solve the following problem of Laplace’s equation for \( u(x, y) \) in a square by eigenfunction expansion:

\[
 u_{xx} + u_{yy} = 0
\]

\[
 u(0, y) = u(1, y) = 2 
\]

\[
 u(x, 0) = u(x, 1) = 4
\]

Use Mathematica or Matlab to make a contour plot of the potential lines by truncating your expansion after some (moderate) number of terms and verifying that your result is insensitive to the number included.
4. Consider a problem of steady convective and diffusive heat transfer across a wide, thin plate that is maintained at a constant (high) temperature $T_0$ and with a (cool) fluid of temperature $T_1$ (before contacting the plate) flowing past its leading edge. With a coordinate system centered at the leading edge of the plate ($x$ streamwise along the plate surface and $y$ normal to the plate), a simple approximation of the streamwise velocity as $v_x = \gamma y$, where $\gamma$ is a constant velocity gradient, and the introduction of a reduced temperature field given by $f(x, y) = \frac{T(x, y) - T_1}{T_0 - T_1}$, the PDE and associated boundary conditions describing this problem can be stated as

$$\frac{\partial f}{\partial x} = \left(\frac{\xi^2}{y}\right)\frac{\partial^2 f}{\partial y^2}, \quad 0 < x, y < \infty$$

$$f(0, y) = 0, \quad y > 0$$

$$f(x, \infty) = 0, \quad x > 0$$

$$f(x, 0) = 1, \quad x > 0$$

where $\xi$ is a constant length dictated by the thermal diffusivity and the velocity gradient, $\xi = (\alpha/\gamma)^{1/2}$. In this exercise we are going to explore a technique referred to as a *similarity solution*, which is related to the method of characteristics, yet works even for non-hyperbolic PDEs in certain circumstances – frequently half-space problems.

(a) Assume a “similarity” solution of the form $f(x, y) = F(\eta)$, with similarity variable $\eta = y/x^p$ and $p$ a real positive exponent. By substitution into the PDE, determine a value of $p$ for the similarity solution to hold and a corresponding second order ODE for the function $F(\eta)$.

(b) Solve the ODE and boundary conditions by a first indefinite integration to solve for $F'(\eta)$ and then a second definite integration from $\eta$ to $\infty$, invoking $F(\infty) = 0$, in order to determine $F(\eta)$ to within a constant. Fix that constant by imposing the second boundary condition $F(0) = 1$. Notice that the similarity solution is enabled by the “compatibility” of the first and second boundary conditions listed above. You may find the following formula useful:

$$\frac{1}{\int_0^\infty d\eta \exp\left[-\eta^2/(9\xi^2)\right]} = 0.538 \xi^{-2/3}$$

(c) Noting that a local heat transfer coefficient $h(x)$ can be defined by the heat flux relation at the plate surface

$$h(x)(T_0 - T_1) = -kT_y(x, 0)$$

where $k$ is the thermal conductivity of the fluid, derive an explicit formula for $h(x)$ that depends only on $x$, $\xi$, and $k$. Can you explain the physical meaning of this expression?