**Operator Treatment of Angular Momentum**

- Work with operators for angular momentum
- Develop using commutation relations
- Don't set up the Schrödinger equation!

**Angular Momentum Operators**

Classical Mechanics

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} i & j & k \\ x & y & z \\ \rho_x & \rho_y & \rho_z \end{vmatrix}
\]

\[
= (yp_z - zp_y)i + (zp_x - xp_z)j + (xp_y - yp_x)k
\]

\[
\Rightarrow \quad L_x = y\rho_z - z\rho_y; \quad L_y = z\rho_x - x\rho_z; \quad L_z = x\rho_y - y\rho_x
\]

Quantum Mechanics - evaluate the commutators:

\[
[L_x, L_y] = [(yp_z - zp_y), (zp_x - xp_z)]
\]

\[
= [yp_z, zp_x] - [yp_z, xp_z] - [zp_x, zp_z] + [zp_z, xp_z]
\]

\[
= y[p_z, z]p_x + p_x[z, p_z]
\]

\[
- i\hbar \quad \Rightarrow \quad i\hbar \quad (xp_y - yp_x) = i\hbar L_z \quad \text{(likewise for other components)}
\]

\[
\Rightarrow \quad [L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y
\]

An observable is an angular momentum if its three components satisfy these commutation relations!

Conclude that the three components of angular momentum cannot be specified simultaneously (since they don't commute). But \([L_z, L^2] = 0\): we can know both the magnitude of angular momentum and one of its components (usually we specify \(L_z\)) simultaneously.
**Shift Operators**

Let's introduce

raising operator \( \hat{L}^+ = \hat{L}_x + i \hat{L}_y \)

lowering operator \( \hat{L}^- = \hat{L}_x - i \hat{L}_y \)

\[
\begin{align*}
\hat{L}_x &= \frac{1}{2}(\hat{L}^+ + \hat{L}^-) \\
\hat{L}_y &= \frac{1}{2i}(\hat{L}^+ - \hat{L}^-)
\end{align*}
\]

Commutation Relations

\[
[\hat{L}^+, \hat{L}^-] = -\hbar \hat{L}^+; \quad [\hat{L}^-, \hat{L}^+] = \hbar \hat{L}^-; \quad [\hat{L}^+, \hat{L}^-] = 2\hbar \hat{L}_z
\]

Also since \( \hat{L}_z \) commutes with all components of \( \hat{L} \), it also commutes with \( \hat{L}^+ \) and \( \hat{L}^- \).

We assume simultaneous eigenstates of \( \hat{L}_z \), \( \hat{L}_x \) are distinguished by two quantum numbers, \( n, m \), so we write the state \( |n, m\rangle \).

Since \( \hat{L}_z \) has the same dimensions as \( \hbar \), its eigenvalues must be some multiple of \( \hbar \), so we define \( m \) through

\[
\{ \hat{L}_z |n, m\rangle = m\hbar |n, m\rangle \}
\]

i.e., \( m\hbar \) is the eigenvalue of \( \hat{L}_z \) for state \( |n, m\rangle \).

We'll see that \( (\hat{L}^-|n, m\rangle) \) is an eigenfn. of \( \hat{L}_z \) as eigenvalue \((m+1)\hbar \), \( (\hat{L}^+|n, m\rangle) \) is an eigenfn. of \( \hat{L}_z \) as eigenvalue \((m-1)\hbar \)

!
Eigenvalues of Angular Momentum

First let's see to what eigenvalue of \( \hat{L}^2 \)
\( \hat{L}^2 |n, m\rangle \) corresponds:

\[
\hat{L}^2 (\hat{L}^+ |n, m\rangle) = \left\{ \hat{L}^+ \hat{L}^2 + \hat{L}^2 \hat{L}^+ \right\} |n, m\rangle
\]
\[
= \{ \hat{L}^+ \hat{L}^2 + \hat{L}^2 \hat{L}^+ \} |n, m\rangle
\]
\[
= \{ \hat{L}^+ \hat{m}_x^2 + \hat{m}_x^2 \hat{L}^+ \} |n, m\rangle
\]
\[
= (m+1) \hbar \{ \hat{L}^+ |n, m\rangle \}
\]

**BUT** \( \hat{L}^2 |n, m+1\rangle = (m+1) \hbar |n, m+1\rangle \)

\[
\therefore \hat{L}^+ |n, m\rangle \propto |n, m+1\rangle
\]

We write \( \hat{L}^+ |n, m\rangle = \hat{c}^+_m |n, m+1\rangle \); \( \hat{L}^- |n, m\rangle = \hat{c}^-_{m-1} |n, m-1\rangle \)
\( \hat{c}^+, \hat{c}^- \) are dimensionless coefficients;
\( \hat{L}^+, \hat{L}^- \) are raising/lowering operators.

Next use the fact that \( \hat{L}^2, \hat{L}^2 \) commute to recognize that \( |n, m\rangle \) is also an eigenstate of \( \hat{L}^2 \), i.e.

\[
\hat{L}^2 |n, m\rangle = \hbar^2 f(n, m) |n, m\rangle
\]

Now the fact that \( \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \) puts a restriction on the relative magnitudes of \( f, m \)

Note that
\[
\langle n, m | \hat{L}^2 | n, m\rangle = \langle n, m | f(n, m) \hbar^2 - m^2 \hbar^2 | n, m\rangle
\]
\[
= \{ f(n, m) - m^2 \hbar^2 \} \tag{1}
\]

and also
\[
\langle n, m | \hat{L}^2 | n, m\rangle = \langle n, m | \hat{L}_x^2 + \hat{L}_y^2 | n, m\rangle
\]
\[
= \langle n, m | \hat{L}_x^2 | n, m\rangle + \langle n, m | \hat{L}_y^2 | n, m\rangle \tag{2}
\]

(since the diagonal elements of the square of any Hermitian operator are non-negative)

\( \therefore \) Comparing (1) and (2) \( \Rightarrow f(n, m) \geq m^2 \)
Now consider effect of $L^+$ on $|\text{n}, \text{m}\rangle$ :-

$$L^+ |\text{n}, \text{m}\rangle = \hat{\text{k}} f(n, \text{m}) |\text{n}, \text{m}\rangle$$

Also $L^+ L \text{n}, \text{m}\rangle = L^2 |\text{n}, \text{m}\rangle$ (since $L^+, L^2$ commute)

$$L^2 |\text{n}, \text{m}\rangle = \hat{k}^2 f(n, \text{m}) L^+ |\text{n}, \text{m}\rangle$$

i.e. $L^+ |\text{n}, \text{m}\rangle$ is an eigenstate of $L^2$ with the same eigenvalue as $|\text{n}, \text{m}\rangle$ itself.

But $L^+$ and $L^2$ ($L^-$ and $L^2$) don’t commute.

$L^+ |\text{n}, \text{m}\rangle$ ($L^- |\text{n}, \text{m}\rangle$) correspond to different eigenvalues of $L^2$. i.e. $L^+, L^-$ shift the value of $m$ but leave the magnitude of momentum unchanged!

![Diagram](image)

**Fig. 6.3.** The effects of the shift operators.

Next use trick of applying $L^+$ to highest state $|\text{n}, \text{M}\rangle$ - this must give 0!

Rearranging $L^2 |\text{n}, \text{m}\rangle = \hat{k}^2 L(L+1) |\text{n}, \text{m}\rangle$

$|\text{n}, \text{m}\rangle = |\text{M}\rangle$

i.e. $\text{m} = \text{L, L-1}....$

as we asserted earlier.
So now we know:

\[ L_z | l, m \rangle = m L | l, m \rangle \]
\[ L^+ | l, m \rangle = \sqrt{l(l+1) - m(m+1)} | l, m+1 \rangle \]
\[ L^- | l, m \rangle = \sqrt{l(l+1) - m(m-1)} | l, m-1 \rangle \]
\[ L^2 | l, m \rangle = l^2 | l, m \rangle \quad m = l, l-1, \ldots \]

We still need to find:
the lower bound for \( m \)
the values of \( c^+, c^- \)
+ to show that \( l, m \) are integers

To find \( c^+, c^- \), we write \( \langle l^- l^+ \rangle \) in terms of:
1. \( l, m \), and
2. \( c^+, c^- \), then equate the two expressions:

\[
\begin{align*}
    c^+_l, m &= \sqrt{l(l+1) - m(m+1)} \\
    c^-_l, m &= \sqrt{l(l+1) - m(m-1)}
\end{align*}
\]

To find the lower bound for \( m \), + to show that \( l, m \) are integers, we apply \( L^- \) to the lowest state, with \( m = M_{\text{min}} \). Then:

\[ L^- | l, m_{\text{min}} \rangle = 0 \]
\[ \therefore \langle l, m_{\text{min}}-1 | L^- | l, m_{\text{min}} \rangle = 0 = c^+_l, m_{\text{min}} \]
\[
\begin{align*}
    c^+_l, m_{\text{min}} &= \sqrt{l(l+1) - m_{\text{min}}(m_{\text{min}}-1)} \\
    \Rightarrow m_{\text{min}} &= -l
\end{align*}
\]

\( \therefore \) the ladder of \( m \) values runs in integer steps from \( m = l \) to \( m = -l \)

\( \therefore \) the ladder is symmetrical \( \therefore l \) must be either integer or half-integers!
**Summary**

- Both the magnitude and z-component of angular momentum are quantized with

\[
|l| = \sqrt{l(l+1)} \hbar , \quad l = 0, \frac{1}{2}, 1, \ldots
\]

\[
|l_z| = m \hbar , \quad m = -l, -l+1, \ldots 0, \ldots l
\]

- Note that applying cyclic boundary conditions to the wavefunctions will restrict \( l \) to integers, but that \( \frac{1}{2} \)-integer angular momentum is allowed. In fact spin angular momentum takes \( \frac{1}{2} \)-integer values.

Usually we use

- \( l, m_l \) for orbital angular momentum
- \( s, m_s \) for spin angular momentum
- \( j, m_j \) for "general" angular momentum.
Eigenfunctions of Orbital Angular Momentum

Let's find the eigenfunction for state $|\ell, \ell\rangle$ - the one with the maximum z component - then generate the others by applying $\hat{\ell}^-$. We can find $|\ell, \ell\rangle$ by solving

$$\hat{l}^+ |\ell, \ell\rangle = 0.$$ 

Use position representation $\Rightarrow$ (in spherical polar coordinates)

$$l_x = i\hbar \left\{ \sin \varphi \frac{\partial}{\partial \varphi} + \cot \theta \cos \varphi \frac{\partial}{\partial \theta} \right\}$$

$$l_y = -i\hbar \left\{ \cos \varphi \frac{\partial}{\partial \varphi} - \cot \theta \sin \varphi \frac{\partial}{\partial \theta} \right\}$$

$$l_z = -i\hbar \frac{\partial}{\partial \varphi}$$

The algebra $\Rightarrow$

$$l^+ = l_x + i l_y = \hbar e^{i\varphi} \left\{ \frac{\partial}{\partial \varphi} + i \cot \theta \frac{\partial}{\partial \theta} \right\}$$

$$l^- = l_x - i l_y = -\hbar e^{-i\varphi} \left\{ \frac{\partial}{\partial \varphi} - i \cot \theta \frac{\partial}{\partial \theta} \right\}$$

Now use $l^+ |\ell, \ell\rangle = 0$

$$\Rightarrow \ h e^{i\varphi} \left\{ \frac{\partial}{\partial \varphi} + i \cot \theta \frac{\partial}{\partial \theta} \right\} \psi_{\ell,\ell}(\theta, \varphi) = 0$$

Separation of variables, $\psi(\theta, \varphi) = \Theta(\theta) \Phi(\varphi)$

$$\Rightarrow \psi_{\ell,\ell}(\theta, \varphi) = N \sin^\ell \theta \ e^{i\varphi}$$

obtain $c = 1$ by requiring $l_z \psi_{\ell,\ell} = \ell \hbar \psi_{\ell,\ell}$

$$\Rightarrow \psi_{\ell,\ell} = N \sin^\ell \theta \ e^{i\varphi}$$

Then use $\hat{\ell}^-$ to generate the fn. corresponding to $m_\ell = L - \ell$.