The One Dimensional Harmonic Oscillator

This is without question the most important one-dimensional problem in wave mechanics. It illustrates many of the qualitative features we have already discussed, and also appears in further theoretical developments throughout quantum theory. The harmonic oscillator Hamiltonian of classical mechanics is:

\[ H_c = \frac{p^2}{2m} + \frac{m \omega^2}{2} q^2 \quad \omega = \frac{k}{\sqrt{m}} \]  \hspace{1cm} (6.87)

and in quantum mechanics it is:

\[ H = \frac{p^2}{2m} + \frac{m \omega^2}{2} q^2 \]  \hspace{1cm} (6.88)

where

\[ [q,p] = i\hbar \]  \hspace{1cm} (6.89)

One can use analytic methods to solve Schrödinger's differential equation and find the wave functions \( \text{as a corollary} \) but it is also instructive to treat the problem algebraically. For this purpose we define new operators:

\[ P = \frac{1}{(m \omega \hbar)^{1/2}} p \quad Q = \left(\frac{m \omega}{\hbar}\right)^{1/2} q \]

Then (6.88) becomes:

\[ H = \frac{\hbar \omega}{2} (P^2 + Q^2) = \hbar \omega H \]  \hspace{1cm} (6.90)

where

\[ H = \frac{1}{2} (P^2 + Q^2) \]  \hspace{1cm} (6.91)

and

\[ [Q,P] = i \]  \hspace{1cm} (6.92)

6.10.1 Eigenvectors and Eigenvalues of \( H \)

Let us define the operators:
\[ a = \frac{1}{\sqrt{2}}(Q + iP) \]
\[ a^+ = \frac{1}{\sqrt{2}}(Q - iP) \]

Note that \( a \) and \( a^+ \) are not hermitian.

From (6.91) and (6.92) we obtain:

\[ [a, a^+] = 1 \] \hspace{1cm} (6.95)

and

\[ H = a^+a + 1/2 \] \hspace{1cm} (6.96)

Consider the operator \( N = a^+a \). Using (6.95) we obtain:

\[ [N, a] = a^+aa - aa^+a = -a \]

and

\[ [N, a^+] = a^+aa^+ - a^+a^+a = a^+ \]

Thus

\[ [a, N] = a \] \hspace{1cm} (6.97)

and

\[ [a^+, N] = -a^+ \] \hspace{1cm} (6.98)

Let \( |x\rangle \) be a non-zero eigenvector of \( N \) with eigenvalue \( x \):

\[ N|x\rangle = x|x\rangle \]

Then, defining \( |v\rangle = a|x\rangle \), we have:

\[ <x|N|x\rangle = <x|a^+a |x\rangle = <v|v\rangle = x<x|x\rangle \]

Thus \( x > 0 \), unless \( |v\rangle = a|x\rangle = 0 \), in which case \( x = 0 \). On the other hand, applying (6.95) to \( |x\rangle \) we obtain:

\[ Na|x\rangle = aN|x\rangle - a|x\rangle = xa|x\rangle - a|x\rangle = (x-1) a|x\rangle \]

which implies that \( a|x\rangle \) is also an eigenvector of \( N \) with eigenvalue \( x-1 \). Furthermore,
\[ Na^+|x> = a^+ N|x> + a^+|x> = xa^+|x> + a^+|x> = (x+1)a^+|x> \]

which implies that \( a^+|x> \) is an eigenvector of \( N \) with eigenvalue \( x+1 \).

We may repeatedly apply \( a \), and generate successive eigenvalues \( x-2, x-3, \ldots \):

<table>
<thead>
<tr>
<th>Eigenvector</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>x&gt; )</td>
</tr>
<tr>
<td>( a</td>
<td>x&gt;</td>
</tr>
<tr>
<td>( a^2</td>
<td>x&gt;</td>
</tr>
</tbody>
</table>

... ... ...

Continuing in this manner we would eventually arrive at negative eigenvalues of \( N \), which is impossible, unless for some vector \( |x_0> \) we have \( a|x_0> = 0 \). This avoids the difficulty because all succeeding applications of \( a \) would just give zero. However, if \( a|x_0> = 0, \) then \( N|x_0> = a^+a|x_0> = 0 \) as well, and \( |x_0> \) is an eigenstate of \( N \) with eigenvalue \( 0 \). Clearly, this implies that the eigenvalues of \( N \) are zero and the positive integers. We thus change our notation, replacing \( |x>, x \) by \( |n>, n \) respectively:

\[ N|n> = n|n> \]

The eigenvalues of the Hamiltonian are:

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega \]  \hspace{1cm} (6.99)

Since these correspond to bound states in a potential that is everywhere finite (except at \( \pm \infty \)) each eigenstate is nondegenerate.

6.10.2 Normalization of Eigenvectors

Let \( |0> \) be the eigenvector corresponding to eigenvalue \( n=0 \), and normalize \( |0> \) to unity: \( <0|0> = 1 \). Also, define:

\[ |1> = a |0> \]

Then,

\[ <1|1> = <0|aa^+ |0> = <0|a^+ a|0> + <0|0> \]

\[ = <0|0> = 1 \]
Next define \( |2\rangle = \frac{1}{\sqrt{2}} \ a^+ |1\rangle \). Then:

\[
\langle 2|2\rangle = \frac{1}{2} (|a^+ a^+ |1\rangle) = \frac{1}{2} (|a^+ a^+ |1\rangle) = \frac{1}{2} (|1\rangle) = \langle 1|1\rangle = 1
\]

Next define \( |3\rangle = \frac{1}{\sqrt{3}} \ a^+ |2\rangle \). By a similar argument it is easy to see that \( \langle 3|3\rangle = 1 \).

In general, we define

\[
|n\rangle = \frac{1}{\sqrt{n!}} a^{|n-1\rangle}
\]

\[
= \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n-1!}} (a^+)^{n-2} |0\rangle
\]

\[
|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^{n} |0\rangle
\]

(6.100)

Similarly, we have:

\[
\frac{a^n}{\sqrt{n!}} |n\rangle = |0\rangle
\]

(6.101)

Thus

\[
\langle n'| N |n\rangle = n \delta_{n',n}
\]

(6.102)

\[
\langle n'| a |n\rangle = n^{1/2} \delta_{n',n-1}
\]

(6.103)

\[
\langle n'| a^+ |n\rangle = (n+1)^{1/2} \delta_{n',n+1}
\]

(6.104)

6.10.3 Wave Function of the Harmonic Oscillator in State \( n \).

Let us find the coordinate representative (wave-function) of \( |n\rangle \). We have \( a|0\rangle = 0 \).

Thus:

\[
\langle Q| Q a 10 \rangle dQ = 0 \quad \text{and} \quad |0\rangle = \int |Q\rangle \langle Q| 10\rangle dQ
\]

(6.105)

Making use of \( a = \frac{1}{\sqrt{2}} (Q + iP) \) and the fact that \( P = -i \ \delta/Q \) in coordinate representation, we find:

\[
\Rightarrow \langle Q'| a 10 \rangle = \int \langle Q'| a |Q\rangle \ u_0(Q) dQ
\]

\[
= \frac{1}{\sqrt{n!}} \int \langle Q'| (Q + iP) \rangle \ u_0(Q) dQ = 0
\]
\[ \int d(Q' - Q) \left( \begin{array}{c} u_0(Q') \\ \frac{\partial}{\partial Q'} \end{array} \right) \cdot \left( \begin{array}{c} v(Q' - Q) \\ u_0(Q') \end{array} \right) \, dQ' = 0 \]

\[ \Rightarrow Q' \frac{\partial}{\partial Q'} u_0(Q') = 0 \]

Therefore

\[ \left( Q + \frac{\partial}{\partial Q} \right) u_0(Q) = 0 \]  \hspace{1cm} (6.106)

The normalized solution to this differential equation:

\[ u_0(Q) = \pi^{-1/4} \exp\left( -\frac{Q^2}{2} \right) \]  \hspace{1cm} (6.107)

is the ground state wave function, a Gaussian centered at the origin. For \( n > 0 \) we have:

\[ u_n(Q) = \langle Q | n \rangle = \langle Q | \frac{(a^+)^n}{n!} | 0 \rangle \]

\[ = \int \left\langle Q \left| \frac{(a^+)^n}{n!} \right| Q' \right\rangle u_0(Q') \, dQ' \]

or:

\[ u_n(Q) = \frac{1}{2^{n/2}n!} \left( Q + \frac{\partial}{\partial Q} \right)^n u_0(Q) \]  \hspace{1cm} (6.108)

In Fig. 6.16, we plot the Harmonic Oscillator wave-functions for the first few values of \( n \).  

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Fig. 6.16 Harmonic oscillator wave functions for \( n = 0 \) to \( n = 4 \).